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Adiabatic Piston as a Dynamical System[1]

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We consider systems of finitely many interacting particles in a cube with a separating wall having a big mass M (adiabatic piston). Assuming that the particles reflect elastically from the ball and the initial velocity of the piston is zero we prove that as M tends to infinity the dynamics of the piston converges to periodic oscillations.

KEY WORDS: Adiabatic piston; adiabatic invariant; averaging method.

Recently E. Lieb attracted the attention of many people to the problem of dynamics of adiabatic piston (see ref. 7). It became so popular that J. Lebowitz even coined the words "notorious piston."

Thermodynamical aspects of the dynamics of the adiabatic piston were considered in many papers. As few examples we can mention[(3, 4, 8)] where the equations of motion of the piston in the thermodynamical limit were derived and analyzed.

In this paper we study the finite-dimensional version of the piston problem assuming that the mass M of the piston tends to infinity and the number of gas particles stays fixed. More precisely, the piston is the separating wall inside a volume $V \subset R^d$, where d is the dimension, $V = \{(x_1,\dots, x_d) \mid a_i \leqslant x_i \leqslant b_i, 1 \leqslant i \leqslant d\}$ and a_i, b_i are fixed. V is cut by the piston Π_X onto two parts $V^{(l)}, V^{(r)}$

$$V^{(l)} = \{(x_1,\dots, x_d) \mid x_1 \leqslant X\}, \qquad V^{(r)} = \{(x_1,\dots, x_d) \mid x_1 \geqslant X\}.$$

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The piston can move along the x_1-axis and changes its velocity under the action of elastic collisions with the "gas" particles inside $V^{(l)}$ and $V^{(r)}$ so that X becomes a function of time. The number and the masses of gas particles inside each part are fixed while the mass M tends to infinity. If initially the velocity $v(0)$ of the piston is zero then the total energy of the system does not depend on M and the absolute value of the piston velocity $v(t)$ at time t is $O(\frac{1}{\sqrt{M}})$. Therefore it is natural to introduce "slow time" $\tau = \frac{t}{\sqrt{M}}$ and study the limiting form of dynamics of the piston in the rescaled time. The main result of this paper states that this limiting dynamics is periodic and its form depends on the Hamiltonians of the left and right particles. For the case of non-interacting particles ("ideal gas") this statement was proven in ref. 9. It turns out that the oscillatory regime of the piston is quite universal and in some sense resembles Carnot cycles.

The Hamiltonian of the whole system "left gas" + "right gas" + piston can be written in the form

$$H = \frac{1}{2}\frac{P^2}{M} + H^{(l)} + H^{(r)}, \tag{1}$$

where $P = Mv$ is the momentum of the piston, $H^{(l)}$, $H^{(r)}$ are Hamiltonians of the left and right particles. To define completely the dynamics we should include "boundary" conditions by assuming elastic collisions of gas particles with the boundary of V and with the moving piston. Formally this can be done by adding the potential equal to zero inside $V^{(l)}, V^{(r)}$ and infinity on the boundary.

As was mentioned above we consider the slow time $\tau = \frac{t}{\sqrt{M}}$. Also we need the limiting dynamics which formally corresponds to $M = \infty$. For this dynamics the piston stays fixed and gas particles undergo elastic collisions with the piston during which the x_1-component of the velocity changes its sign but the others remain unchanged.

The microcanonical distribution $\frac{d\sigma_\alpha}{|\text{grad } H^{(\alpha)}|}$, $\alpha = l$ or r, is an invariant measure for the α-part concentrated on the manifold $\Sigma^{(\alpha)}$ of constant energy $H^{(\alpha)} = h$ provided that the piston is fixed. We shall use phase averages:

$$\langle f \rangle_\alpha = \frac{1}{\displaystyle\int_{\Sigma^{(\alpha)}} \frac{d\sigma_\alpha}{\text{grad } H^{(\alpha)}}} \int_{\Sigma^{(\alpha)}} \frac{f(x)\, d\sigma_\alpha}{|\text{grad } H^{(\alpha)}|}.$$

Basic Assumption. For almost every values of the energies $h^{(l)}$ of $H^{(l)}$ and $h^{(r)}$ of $H^{(r)}$ and almost every position of the piston X the dynamics on each $\Sigma^{(\alpha)}$ is ergodic wrt the microcanonical distribution.

Under this assumption we can use averaging method. Denote by $I^{(\alpha)}=I^{(\alpha)}(h,X)$, $\alpha=l,r$ the phase volume of the set $\{H^{(\alpha)}<h\}$ provided that the position of the piston is X. It is easy to derive the formulas (see ref. 6)

$$\frac{\partial I^{(\alpha)}}{\partial h}=\int_{\Sigma^{(\alpha)}}\frac{1}{|\mathrm{grad}\ \ H^{(\alpha)}|}\,d\sigma_\alpha, \tag{2}$$

$$\frac{\partial I^{(\alpha)}}{\partial X}=-\int_{\Sigma^{(\alpha)}}\frac{\partial H^{(\alpha)}}{\partial X}\,\frac{d\sigma_\alpha}{|\mathrm{grad}\ \ H^{(\alpha)}|}. \tag{3}$$

Strictly speaking the last formula has a well defined meaning when the hard-core potential at the piston is replaced by a "soft potential." The limiting form as the soft potential tends to the hard-core is a δ-function integral over a submanifold of $\Sigma^{(\alpha)}$ where the coordinate of one of the particles coincides with the coordinate of the piston. We do not discuss this in more detail and only remark that from (2) and (3)

$$\frac{\partial I^{(\alpha)}}{\partial X}=-\frac{\partial I^{(\alpha)}}{\partial h}\left\langle\frac{\partial H^{(\alpha)}}{\partial X}\right\rangle_\alpha. \tag{4}$$

Denote $\bar{P}=P/\sqrt{M}$, $\varepsilon=\frac{1}{\sqrt{M}}$. Then

$$H=\tfrac{1}{2}\bar{P}^2+H^{(l)}+H^{(r)}$$

and we can write down the equations of motion

$$\dot{X}=\varepsilon\bar{P}, \tag{5$'$}$$

$$\dot{\bar{P}}=-\varepsilon\left(\frac{\partial H^{(l)}}{\partial X}+\frac{\partial H^{(r)}}{\partial X}\right), \tag{5$''$}$$

$$\frac{dH^{(l)}}{dt}=\varepsilon\,\frac{\partial H^{(l)}}{\partial X}\,\bar{P}, \tag{5$'''$}$$

$$\frac{dH^{(r)}}{dt}=\varepsilon\,\frac{\partial H^{(r)}}{\partial X}\,\bar{P}. \tag{5$''''$}$$

In the last two equations $H^{(l)}, H^{(r)}$ are considered as functions of phase coordinates of gas particles and of the position of the piston. Since gas particles interact with the piston $H^{(\alpha)}$ are not the first integrals of the exact dynamics.

The limiting dynamics corresponding to $M=\infty$ is ergodic. Therefore, using the averaging method, we can replace Eqs. (5′)–(5′′′′) by the averaged equations of motion:

$$\dot{X}=\varepsilon\bar{P}, \tag{6'}$$

$$\dot{\bar{P}}=-\varepsilon\left(\left\langle\frac{\partial H^{(l)}}{\partial X}\right\rangle_l+\left\langle\frac{\partial H^{(r)}}{\partial X}\right\rangle_r\right), \tag{6''}$$

$$\frac{dH^{(l)}}{dt}=\varepsilon\left\langle\frac{\partial H^{(l)}}{\partial X}\right\rangle_l\bar{P}, \tag{6'''}$$

$$\frac{dH^{(r)}}{dt}=\varepsilon\left\langle\frac{\partial H^{(r)}}{\partial X}\right\rangle_r\bar{P}. \tag{6''''}$$

The right-hand sides of these equations depend on X, $\bar{P}$ and the values of $H^{(l)}$, $H^{(r)}$. Thus in slow time we get a closed system of equations for these variables. The theorem by Anosov (see ref. 1) says that on time intervals $O(\frac{1}{\varepsilon})=O(\sqrt{M})$ the measure of the set of initial phase points where solutions of the exact equations are arbitrarily close to solutions of the averaged equations tends to 1 as $M\to\infty$.

Remark. Anosov theorem was proved for system with smooth Hamiltonians. It is possible to check that it remains valid in our case as well.

Return back to the variables $I^{(\alpha)}$. Differentiation of these variables along solutions of the averaged systems gives (see (4))

$$\dot{I}^{(\alpha)}=\varepsilon\left(\frac{\partial I^{(\alpha)}}{\partial H^{(\alpha)}}\left\langle\frac{\partial H^{(\alpha)}}{\partial X}\right\rangle_\alpha+\frac{\partial I^{(\alpha)}}{\partial X}\right)=0.$$

Therefore $I^{(\alpha)}(H^{(\alpha)},X)$ are the first integrals of the averaged system. By this reason they are approximately conserved for majority of initial conditions by the exact system on time intervals $O(\sqrt{M})$ and are called sometimes almost adiabatic invariants (see ref. 2).

Let us consider again the functions $I^{(\alpha)}(H^{(\alpha)},X)$. Assume that they are non-degenerate and we can invert them to write $H^{(\alpha)}=F^{(\alpha)}(X,I^{(\alpha)})$, $\alpha=l,r$. Equation (4) implies that $\partial F^{(\alpha)}/\partial X=\langle\partial H^{(\alpha)}/\partial X\rangle_\alpha$.

Introduce the effective Hamiltonian for the piston

$$\mathscr{H}(\bar{P},X,I^{(l)},I^{(r)})=\tfrac{1}{2}\bar{P}^2+W(X,I^{(l)},I^{(r)}),$$

where $W(X,I^{(l)},I^{(r)})=F^{(l)}(X,I^{(l)})+F^{(r)}(X,I^{(r)})$.

As was said above, $I^{(l)}$ and $I^{(r)}$ are almost adiabatic invariants and $\mathscr{H}$ gives the Hamiltonian of the limiting dynamics of the piston.

Consider several examples.

1. Assume that the gases consist of hard balls or disks in the two-dimensional case of the same radius (see ref. 5). Then $I^{(\alpha)}(H, X) = C^{(\alpha)} H^{\frac{d}{2} n^{(\alpha)}} Q^{(\alpha)}(X)$, where $Q^{(\alpha)}(X)$ is the partition function corresponding to the system of $n^{(\alpha)}$ particles, $C^{(\alpha)}$ is an absolute constant depending on $n^{(\alpha)}$. Thus

$$H = ((Q^{(\alpha)}(X)\, C^{(\alpha)})^{-1}\, I^{(\alpha)})^{\frac{2}{dn^{(\alpha)}}}$$

and the potential W takes the form

$$W = ((C^{(l)})^{-1}\, I^{(l)})^{\frac{2}{dn^{(l)}}} (Q^{(l)}(X))^{-\frac{2}{dn^{(l)}}} + ((C^{(r)})^{-1}\, I^{(r)})^{\frac{2}{dn^{(r)}}} (Q^{(r)}(X))^{-\frac{2}{dn^{(r)}}}.$$

If we write formally $Q^{(\alpha)}(X) = e^{n^{(\alpha)} f^{(\alpha)}(x)}$, where $f^{(\alpha)}(x)$ is proportional to the free energy, then

$$(Q^{(\alpha)}(X))^{\frac{2}{dn^{(\alpha)}}} = e^{\frac{2}{d} f^{(\alpha)}(X)}.$$

2. In the cases of the ideal gases the dynamics of left and right particles are non-ergodic on the manifold of constant energy. However, if we fix all additional first integrals we get the effective potential $W = \frac{C_l}{X^2} + \frac{C_r}{(L-X)^2}$, $a_1 = 0$, $L = b_1$.

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